# Stability criteria

If a system has transfer function $G(s) = \frac{b(s)}{a(s)}$ where $b(s)$ and $a(s)$ are polynomials, $a(s)$ is called the characteristic polynomial. Its roots are the poles of the system. These poles determine stability.

A system is stable if all of its poles are in the left-half plane. That is, whenever $a(s) = 0$, we must have $Re(s) < 0$.

It's always possible to check stability by:

(i) calculating all roots of the characteristic polynomial
(ii) checking that each root has negative real part.

But if we don't care about individual poles and we just want to know if the system is stable, there is a faster way!

First, normalize the transfer function so the leading coefficient of $a(s)$ is 1. For example:

$$\frac{3s+7}{-2s^2+5s+1} = \frac{-\frac{3}{2}s - \frac{7}{2}}{\boxed{s^2 - \frac{5}{2}s - \frac{1}{2}}}$$

normalized characteristic polynomial.

$\boxed{\text{Fact}}$: <u>if</u> the system is stable, <u>then</u> <u>all</u> coefficients of the normalized characteristic polynomial are <u>positive</u>.

This is because if $a(s)$ has all roots in the left-half plane, it must look like:

$$a(s) = \underbrace{(s+r_1)(s+r_2)\cdots(s+r_k)}_{\text{real roots}} \underbrace{((s+p_1)^2+q_1^2)\cdots((s+p_\ell)^2+q_\ell^2)}_{\text{complex roots}}$$

where $r_i$, $p_i$ are positive. Multiplying this out must yield a polynomial with all coefficients positive since there is no way to make a negative number!

The converse is <u>not</u> true. i.e. just because all coefficients are positive, that doesn't necessarily mean all roots are in the left-half plane. For example:

$$\underbrace{s^3 + s^2 + 4s + 30}_{\text{all coefficients positive}} = \underbrace{(s+3)(s^2-2s+10)}_{\text{roots at: } -3,\ 1\pm 3j}$$

[ Why is this useful? We can <u>quickly</u> rule out stability when there are <u>any</u> negative coefficients! (or even any zero coefficients) ]

In math-speak, we say that: "all coefficients being positive is a <u>necessary</u> condition for stability, but it is <u>not</u> <u>sufficient</u>".

* Warning: be sure to normalize first! Otherwise replace "all coefficients positive" by: "all coefficients positive <u>or</u> all coefficients negative".

There are more sophisticated tests that tell you precisely when you have stability (i.e. conditions that are necessary and sufficient).

Here are the conditions for low-order characteristic polynomials:

First order: $a(s) = s + a_0$.

stable if and only if (iff) $a_0 > 0$.

2nd order: $a(s) = s^2 + a_1 s + a_0$.

stable iff $a_0 > 0$ and $a_1 > 0$.

3rd order: $a(s) = s^3 + a_2 s^2 + a_1 s + a_0$.

stable iff $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, and $a_1 a_2 > a_0$.

4th order: $a(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$.

stable iff $a_0 > 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1 a_2 a_3 > a_1^2 - a_0 a_3$.

5th order and above: there are conditions, but they are increasingly more complicated. For more information about this, you can look up the "Routh-Hurwitz stability criterion".

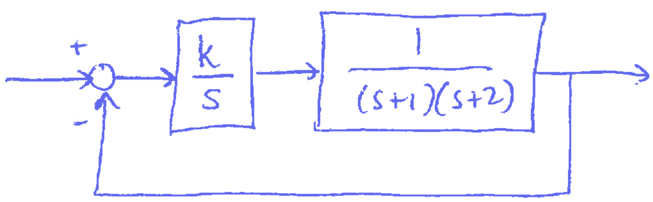
## Examples

1) Is this system stable? $\dfrac{s-2}{s^2+7s+34}$.

Yes. (it's 2nd order so we only have to check positivity of the denominator coefficients; $7>0$ and $34>0$.).

2) Are all the roots of $s^3+s^2+4s+30$ in the left-half plane?
[this was the example on page 2]. $a_2=1>0$, $a_1=4>0$, $a_0=30>0$.
But the final condition: $a_1 a_2 > a_0$ fails! $4 \not> 30$. So the system is unstable. (at least one root in the right-half plane).

3) We saw that intuitively, I-control could cause oscillation and potentially instability. Lets give a concrete example.




For what values of $k$ is this system + compensator stable?

The closed-loop map is: $\dfrac{C\cdot G}{1+C\cdot G} = \dfrac{\frac{k}{s}\cdot\frac{1}{(s+1)(s+2)}}{1+\frac{k}{s}\cdot\frac{1}{(s+1)(s+2)}} = \dfrac{k}{s^3+3s^2+2s+k}$

For this third-order characteristic polynomial,
→ all coefficients must be positive, so $k>0$.
→ $a_1 a_2 > a_0$, so $2\cdot 3 > k$, so $k<6$.

the system is stable for $0<k<6$.